Math 105 Chapter 2: Functions of Several Variables.

In math 104, we studied functions of the form

$$y = f(x).$$

What that says is, given an x-value, we determine a unique y value given by some "rule", that "rule" is called a function f.

eg $y = x^2$, we have $f(x) = x^2$, meaning given an x, we square it to get the y-value.

Recall that $f(x)$ usually represented some physical quantity that we wanted to model that depends on x.

eg The price P, depends on the quantity q of goods produced so we have P is a function of q, denoted by $P(q)$.

Now notice in 104 we always assumed that whatever quantity we were modeling depended only on one quantity, and if it didn't, there was some magic constraint which made it depend on one variable. Now in real life, as we model more complex things it is unrealistic to think a quantity only depends on one factor.

eg. The revenue of a company can depend on many factors such The amount of products it sells, the size of the market share it has, the demand of their product, time, etc...

To talk about such things we need define functions of several variables.

Definition: A function of n-variables is a rule f that takes points in $\mathbb{R}^n$ and assigns unique value to them. They are denoted by

$$f(\vec{x}) = f(x_1, .., x_n)$$

Eg $f(x,y) = xy^2 + 1$

$f(x,y) = \ln x$, not every function needs to have a term with y.

$f(x,y) = 2$, not every function needs to have a term with x,y.

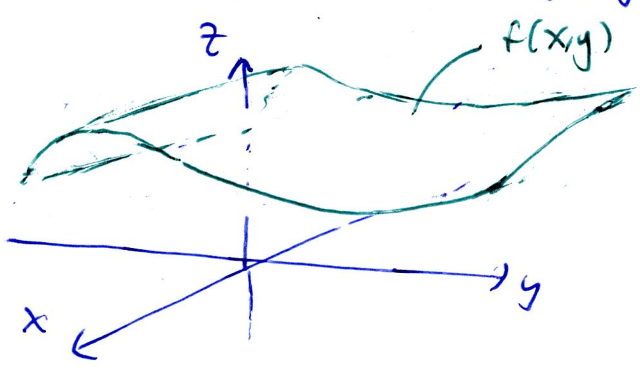
From now on we will only be dealing with functions of 2 variables since as you will soon see, 2 variables will make our lives interesting enough.

We will be spending the next few weeks generalizing all concepts we built up for 1 variable functions to 2 and higher. It will generalize nicely but will give us a few headaches, so have that tylenol ready. (Note: I do not condone the use of drugs (for this course), if you actually have a headache, get some sleep or drink water).

We will also be using the notation

$$z = f(x,y)$$

To denote the unique value z given a point x,y, eg.



Definition: Given a function $z = f(x,y)$ we define the Domain of f by:

$$D(f) = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) \text{ exists} \}$$

The range of f is

$$R(f) = \{ c \in \mathbb{R} \mid f(x,y) = c \text{ for some } (x,y) \in D(f) \}$$

A graph of f is the set of points in $\mathbb{R}^3$ of the form:

$$(x, y, f(x,y)).$$

That is a lot of techical jargon, but what it means is that the domain is the point that the function is defined. The range is the possiple z-values a function can take on and the graph is What you visualiz when you think of f.

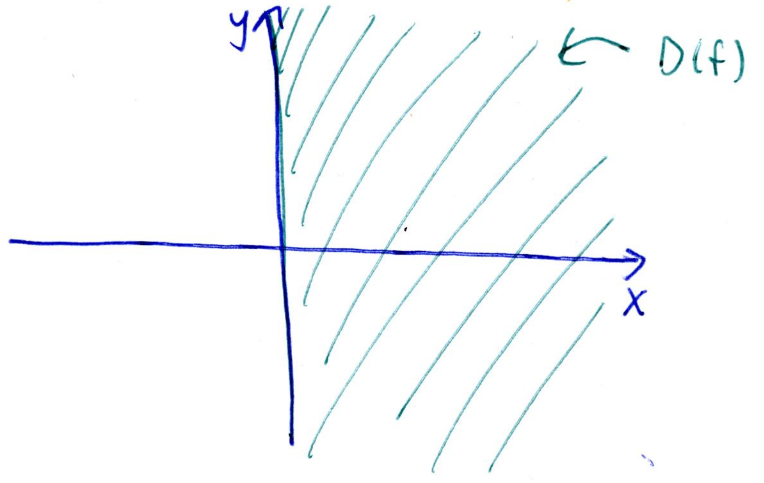
eg What is the domain of

a) $f(x,y) = \sin(\sqrt{x}\, y^3)$

b) $g(x,y) = \dfrac{1}{9 - x^2 - 4y^2}$

a) Note $\sin(x)$ is defined for all x, so $\sin(\sqrt{x}\, y^3)$ is defined whenever $\sqrt{x}\, y^3$ is. y^3 is defined for all y, but $\sqrt{x}$ is only defined for $x \geq 0$.

$$\Rightarrow \quad D(f) = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0 \}$$



The shaded region is the domain of f.

b) $9-x^2-4y^2$ is defined for all x, y so the only time g is undefined is when the denominator is zero, that happen when

$$9-x^2-4y^2=0$$

$$\Rightarrow \quad x^2+4y^2=9$$

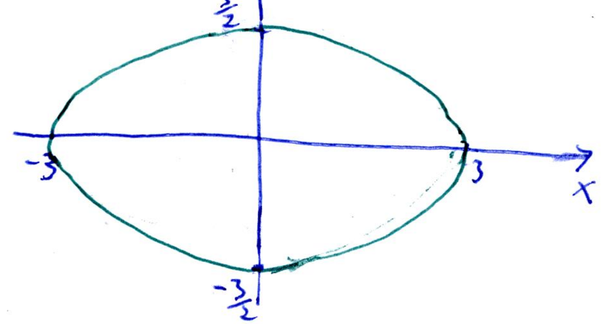
This is an equation of an ellipse, centered at $(0,0)$. Now how do we draw this? Let know the ellipse is symetric about the x, and y-axis. and x^2 is a maximum when $y=0$, ie

$$x^2=9 \Rightarrow x= \pm 3$$

y^2 is a maximum when $x=0$, ie

$$4y^2=9 \Rightarrow y= \pm \frac{3}{2}$$

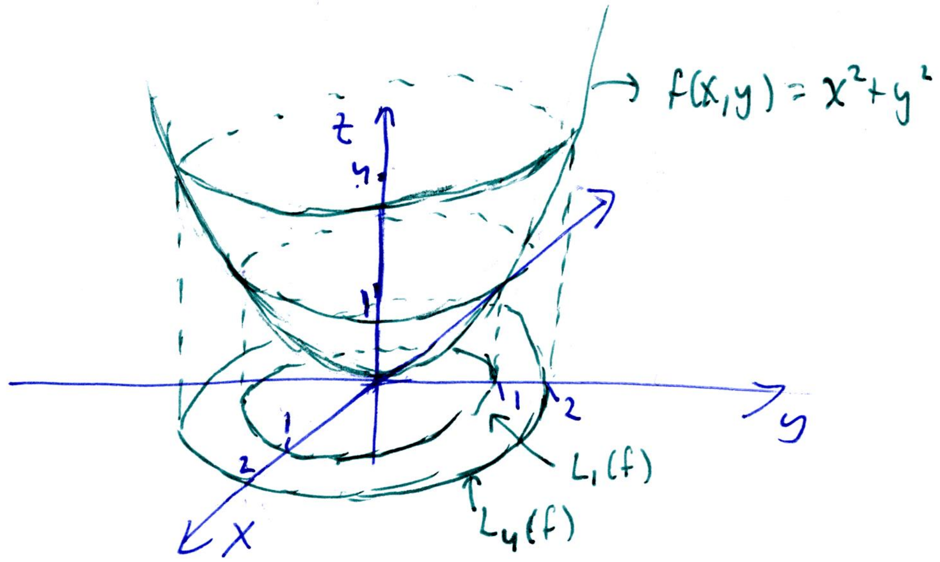
So the domain is the x-y plane without the ellipse.



Definition: A level set or contour of $f(x,y)$ for a constant c is the set of points $(x,y) \in \mathbb{R}^2$ such that $f(x,y)=c$.
I will denote it as:

$$L_c(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y)=c\}$$

eg.



The graph above is a paraboid given by the equation

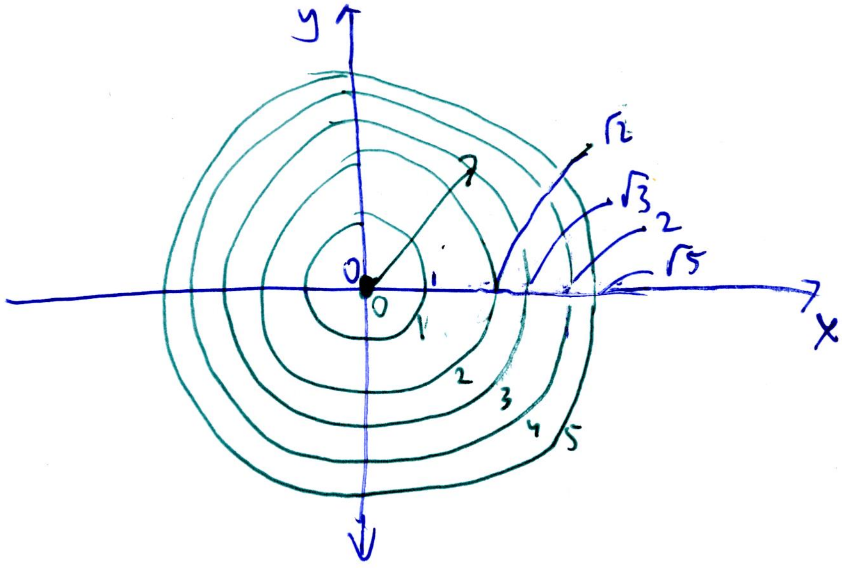
$$z = f(x,y) = x^2+y^2$$

When is $f(x,y)=c$? Lets see

$$c = f(x,y) = x^2+y^2$$

So if $c<0$, no x,y will make that true since $f \geq 0$.
$c=0$, then $f(x,y)=0$ only if $x=y=0$
$c>0$, then $x^2+y^2=c$, which are the points on the circle of radius $\sqrt{c}$

So $L_c(f)$ is nothing if $c<0$
is the origin if $c=0$
is the circle of radius $\sqrt{c}$ if $c>0$.



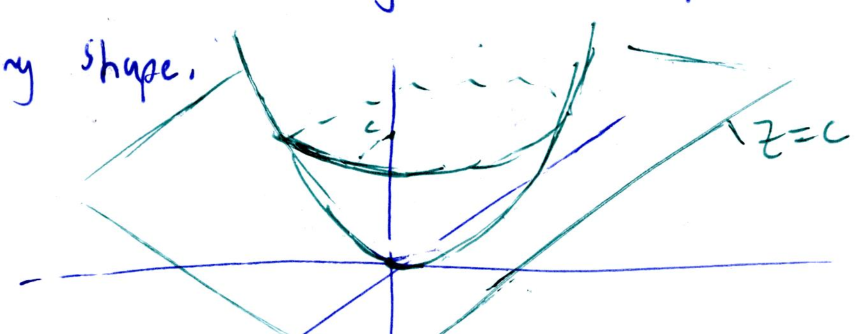
My poorly drawn contour plot does not get across as well as I would like to that the circles are concentric and get closer to eachother. The green values indicate the level set the circles represent and the arrow signifies that the function increases along the arrow.

Contours are often used to sumarize how a function behaves without actually graphing.

<u>Excercise</u>: Given $a, b > 0$ find contour plots of

$$f(x,y) = a^2x^2 + b^2y^2$$
$$f(x,y) = a^2x^2 - b^2y^2$$

One way you can think of a contour is by taking the graph of your function and slicing it by a plane $z=c$ and seeing the resulting shape.

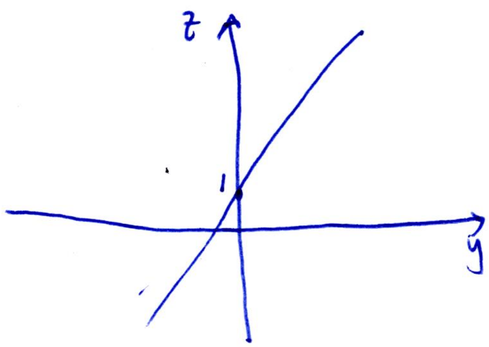


In general to find the trace of $x=c$ of a function, you take the x-variable, replace it with c and graph the resulting function:

eg $\quad z = x^2y + 2xy + 1$

- The trace of $x=2$ is given by

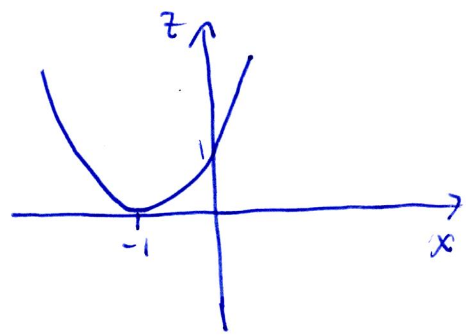
$$z = 2^2y + 2(2)y + 1$$
$$= 8y + 1$$



Which is a line in the yz-plane

- The trace of $y=1$ is given by

$$z = x^2 + 2x + 1$$
$$= (x+1)^2$$



Which is a quadric in the xz-plane

Partial Derivatives!

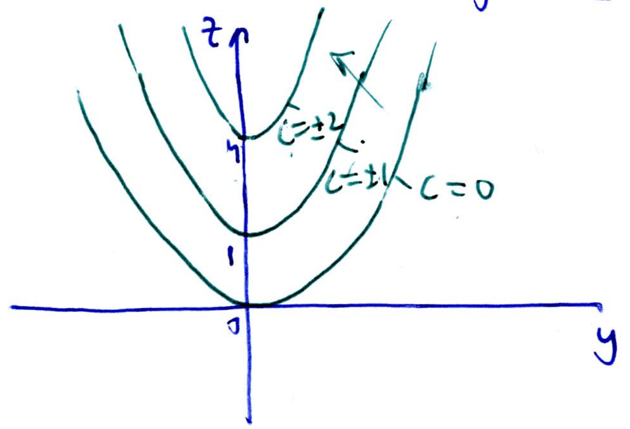
Now we will switch gears and begin our discussion of derivatives. Recall from calc I, that we used derivatives to represent how a function $f(x)$ changed given a "small" change in x. We want to translate that idea for function of the form $f(x,y)$. How do we quantify change when we have 2 variables to keep track of. Simple: we just make one of them change.

There is nothing special about the plane $z=c$ what happens when you cut the graph through a diffifert plane?

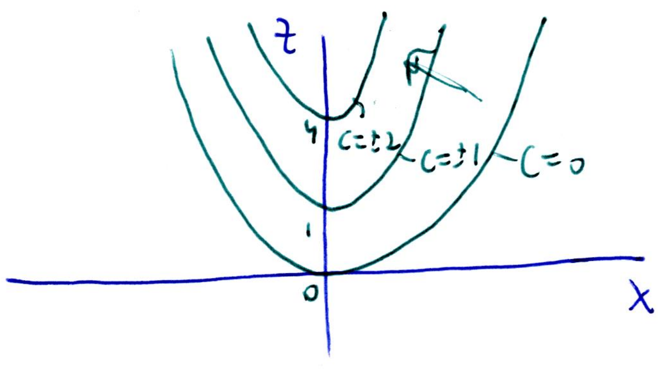
<u>Definition</u>: A <u>trace</u> of a function $z=f(x,y)$ is the resulting shape when a function intersects a plane.

eg Let us comput the trace of the paraboloid, with $x=c$, $y=c$.

When $x=c$, we have $z=c^2+y^2$ so on the yz-plane we get



Similarly $y=c$, we have $z=x^2+c^2$ and on the xz-plane



Definition: Given a function $f(x,y)$, partial derivative of f with respect to x is the rate of change of f in the x with y held fixed. It is denoted as

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x$$

We can define the partial derivative of f in the y direction analogously, and is denoted as

$$\frac{\partial f}{\partial y} \quad \text{or} \quad f_y$$

Eg In a macroeconomic model the GDP (Y) is a function of investment income (I), consumption (C), government spending (G) and net exports (X).

$$Y(I, C, G, X)$$

$\frac{\partial Y}{\partial I}$ represents the rate of change of GDP if we vary Investment income but leave consumption, government spending and net exports fixed.

Definition: A more concrete (but equivalent) definition of partial derivatives is in terms of limits.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

To comute f_x we predend y is a constant and take derivative with x from calc 1.

eg a) $f(x,y) = x^4 + x^3 + 2$

b) $f(x,y) = \log(xy + \sin y) + \tan y$

c) $f(x,y) = x^y + \pi$

Find f_x, f_y:

a) $f_x = 4x^3 + 3x^2$

$f_y = 0$, since f does not depend on y.

b) $f_x = \dfrac{1}{xy + \sin y} \cdot y$

$f_y = \dfrac{1}{xy + \sin y} \cdot (x + \cos y) + \sec^2 y$

c) $\dfrac{\partial f}{\partial x} = y\, x^{y-1}$, since $\dfrac{d}{dx} x^a = a x^{a-1}$

$\dfrac{\partial f}{\partial y} = x^y \log x$, since $\dfrac{d}{dx} a^x = a^x \log a$

What about the analog of the second derivative, now that we have 2 variable we have more second order derivatives:

Definition: The second order partials are defined by:

$f_{xx} = \dfrac{\partial}{\partial x} f_x = \dfrac{\partial^2}{\partial x^2} f$, $f_{yy} = \dfrac{\partial}{\partial y} f_y = \dfrac{\partial^2}{\partial y^2} f$

$f_{xy} = \dfrac{\partial}{\partial y} f_x = \dfrac{\partial^2}{\partial y \partial x} f$, $f_{yx} = \dfrac{\partial}{\partial x} f_y = \dfrac{\partial^2}{\partial x \partial y} f$

We can define f_{xxx}, f_{xyx}, ... respectively.

eg: $f(x,y) = x^3y + yx + y^2 + x$

- $f_x = 3x^2y + y + 1$
- $f_{xx} = 6xy$
- $f_{xy} = 3x^2 + 1$
- $f_y = x^3 + x + 2y$
- $f_{yy} = 2$
- $f_{yx} = 3x^2 + 1$

Note $f_{xy} = f_{yx}$, but is this always true?

Theorem: (Clairot's theorem) Suppose that f_x and f_y continuous and defined on the domain of f, then

$$f_{xy} = f_{yx}$$

eg. Is there a function with $f_x = 2y$, and $f_y = 3x$?

By Clairot's theorem since f_x and f_y are continuous, if such a function was to exist, then $f_{xy} = f_{yx}$; But,

$$f_{xy} = 2 \neq 3 = f_{yx}$$

So no such f exists.

Note the fact that f_x, f_y are continuous is essential.

eg. $f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}$

$$f_x(x,y) = \frac{3x^2 - y^3}{x^2+y^2} - \frac{2x(x^3y - xy^3)}{(x^2+y^2)^2}$$

$$\Rightarrow f_x(0,y) = -y$$

$$\Rightarrow f_y(0,0) = -1$$

$$f_y(x,y) = \frac{(x^3 - 3xy^2)}{x^2+y^2} - \frac{2y(x^3y - xy^3)}{(x^2+y^2)^2}$$

$$\Rightarrow f_y(x,0) = x$$

$$\Rightarrow f_{xy}(0,0) = 1$$

So $f_{xy}(0,0) \neq f_{yx}(0,0) \Rightarrow f_{xy} \neq f_{yx}$

<u>Definition:</u> The gradient of a function is $f(x,y)$ is:

$$\nabla f = (f_x, f_y)$$

The gradient summarizes the derivatives of a function and is incredibly important for the study of functions of multiple variable but we will only limited results.